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Weighted Composition Operators from Fractional Cauchy Transforms to Logarithmic Weighted - Type Spaces

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Abstract: In this paper, we consider weighted composition operators from fractional Cauchy transforms to logarithmic weighted- type spaces.Upper and lower bounds for norm of these operators are computed and compactness is completely characterized.

Keywords: Differentiation operator; composition operator; fractional Cauchy transforms; logarithmic weighted – type spaces.

1. Introduction and Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\boldsymbol{T}$ its boundary, $dA(z)\frac{1}{\pi}\,dxdy = \frac{1}{\pi}\,rdrd\theta$the normalized area measure on$\mathbb{D}$,H^{∞} the space of all bounded holomorphic functions on $\mathbb{D}$ with the norm $||f||_{\infty} = sup_{z\in\mathbb{D}}|f(z)|$, $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$.

Let

$\gamma_a(z) = \frac{(a-z)}{(1-\bar{a}z)}$, $a, z \in \mathbb{D}$,

That is, the involutive automorphism of $\mathbb{D}$ interchanging points a and 0.

For $\alpha > 0$, the family K_α of fractional Cauchy transforms is the collection of functions $f \in H(\mathbb{D})$ which are represented as

$$f(z) = \int_{\boldsymbol{T}} \frac{1}{(1-\bar{\zeta}z)^{\alpha}} d\mu(\zeta)(z \in \mathbb{D}). \quad (1.1)$$

for some $\mu \in \mathcal{M}$, the space of all complex Borel measure on $\boldsymbol{T}$. The principal branch is used in the power function in (1.1) and throughout the rest of the paper. The space K_α is a Banach space with respect to the norm

$$||f||_{K_\alpha} = inf_{\mu\in\mathcal{M}}\{||\mu||: f(z) \text{is given by } (1.1)\}, \quad (1.2)$$

where $\left||\mu|\right|$ denotes the total variation of measure μ. The space K_α may also be written as $K_\alpha = (K_\alpha)_a + (K_\alpha)_s$, where $(K_\alpha)_a$ is isometrically isomorphic to $\mathcal{M}/\overline{H}_0^1$, the closed subspace of $\mathcal{M}$ of absolutely continuous measure and $(K_\alpha)_s$ is isomorphic to $\mathcal{M}_s$,the closed subspace of $\mathcal{M}$ of singular measures. Moreover, for $f \in K_\alpha$,

$$|f(z)| \leq \left||f|\right|_{K_\alpha} / \ (1-|z|)^{\alpha} (z \in \mathbb{D}). \quad (1.3)$$

For more about these spaces see [1],[2], [3], [4], [6], [7], [8], [9], [10].

The logarithmic weighted space $A^{\beta}_{ln}(\mathbb{D}) = A^{\beta}_{ln}$ consists of all $f \in H(\mathbb{D})$ such that

$$||f||_{A^{\beta}_{ln}} = sup_{z \in \mathbb{D}}(1-|z|^2)^{\alpha}|f(z)|ln\frac{2}{1-|z|^2}<\infty.$$

With the norm $||.||_{A^{\beta}_{ln}}$, the space A^{β}_{ln} is a Banach space.

Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. For a non-negative integer n, we define a linear operator $W^{n}_{\psi,\varphi}$ as follows:

$$W^{n}_{\psi,\varphi} f = \psi . f^{(n)} \, o \, \varphi$$

for $f \in H(\mathbb{D})$.The operator $W^{n}_{\psi,\varphi}$ is called a weighted composition operator.

It is of interest to provide function- theoretic characterization of boundedness and compactness of $W^{n}_{\psi,\varphi}$ from the space of fractional Cauchy transforms to different spaces of holomorphic functions. For some recent results in this area, see [11],[12], [13], and the references therein. In this paper, we characterize boundedness and compactness of weighted composition operators from fractional Cauchy transforms to logarithmic weighted - type spaces. Throughout the paper constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notation $A \asymp B$ means there is a positive constant C such that $A/C \le B \le CA$.

2. Boundedness and compactness of $W^{n}_{\psi,\varphi} : K_{\alpha} \rightarrow A^{\beta}_{ln}$

In this section, we characterize the boundedness and compactness of $W^{n}_{\psi,\varphi}$ from the space of fractional Cauchy transforms to logarithmic weighted - type spaces.

Theorem 1. Let $\alpha > 0$, $\beta > 0, n \in N \cup \{0\}, \psi \in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$. Then $W^{n}_{\psi,\varphi} : K_{\alpha} \rightarrow A^{\beta}_{ln}$ is bounded if and only if

$$M_1 := sup_{\zeta \in \boldsymbol{T}} sup_{z \in \mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{|1-\bar{\zeta}\,(z)|^{n+\alpha}}|\psi(z)|ln\frac{2}{1-|z|^2} < \infty. \tag{2.1}$$

Proof: First suppose that (2.1) holds. Let $f \in K_{\alpha}$. Then there is a $\mu \in \mathcal{M}$ such that $||\mu|| = ||f||_{K_{\alpha}}$ and

$$f(z) = \int_{\boldsymbol{T}} \frac{1}{(1-\bar{\zeta}z)^{\alpha}} d\mu(\zeta)$$

Thus, we have

$$f^{n}(z) = \alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1)\int_{\boldsymbol{T}} \frac{(\bar{\zeta})^{n}}{(1-\bar{\zeta}z)^{n+\alpha}} d\mu(\zeta). \tag{2.2}$$

Replacing z in (2.2) by $\varphi(z)$, using a known inequality and multiplying such obtained inequality by

$(1-|z|^2)^{\alpha} ln\frac{2}{1-|z|^2}|\psi(z)|$, we obtain

$$|f^{n}\varphi(z)| \le \alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1)\int_{\boldsymbol{T}} \frac{(1-|z|^2)^{\alpha}}{|1-\bar{\zeta}\,(z)|^{n+\alpha}}|\psi(z)|ln\frac{2}{1-|z|^2} d|\mu|(\zeta) \tag{2.3}$$

$$\le \alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1) sup_{\zeta \in \boldsymbol{T}} sup_{z \in \mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{|1-\bar{\zeta}\,(z)|^{n+\alpha}}|\psi(z)|ln\frac{2}{1-|z|^2}\int_{\boldsymbol{T}} d|\mu|(\zeta)$$

$= \alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1) sup_{\zeta\in \boldsymbol{T}} sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{|1-\bar{\zeta}\,(z)|^{n+\alpha}} |\psi(z)| ln\frac{2}{1-|z|^2} ||\mu||$

from which it follows that

$(1-|z|^2)^{\alpha} |\psi(z)| ln\frac{2}{1-|z|^2} |f^n\,\varphi(z)| \leq$

$\alpha(\alpha+1)(\alpha+2)..(\alpha+n-1) sup_{\zeta\in \boldsymbol{T}} sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{|1-\bar{\zeta}\,(z)|^{n+\alpha}} |\psi(z)| ln\frac{2}{1-|z|^2} ||f||_{K_\alpha}.$

Taking the supremum over $z\in\mathbb{D}$**,** we get

$$||W^n_{\psi,\varphi} f||_{A^{\alpha}_{ln}} = sup_{z\in\mathbb{D}} |(W^n_{\psi,\varphi} f)(z)| \leq \alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1) M_1 \, ||f||_{K_\alpha}. \quad (2.4)$$

Next suppose that $W^n_{\psi,\varphi} : K_\alpha \rightarrow A^{\beta}_{ln}$ is bounded.

Let

$$f_\zeta(z) = \int \boldsymbol{T} \frac{1}{(1-\bar{\zeta}z)^{\alpha}} d\mu(\zeta), \ \zeta \in \boldsymbol{T}. \quad (2.5)$$

Then $||f_\zeta||_{k_\alpha} = 1$ and

$f^n_\zeta(z) = \alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1) \frac{(\bar{\zeta})^{n}}{(1-\bar{\zeta}(z)^{n+\alpha}}.$

From this and the boundedness of the operator $W^n_{\psi,\varphi} : K_\alpha \rightarrow A^{\beta}_{ln}$, we have that $||W^n_{\psi,\varphi} f_\zeta||_{A^{\beta}_{ln}} \leq ||W^n_{\psi,\varphi}||_{K_\alpha \rightarrow A^{\beta}_{ln}}$, for every $\zeta \in \boldsymbol{T}$ and so

$\alpha(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n-1) sup_{\zeta\in \boldsymbol{T}} sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{|1-\bar{\zeta}\,(z)|^{n+\alpha}} |\psi(z)| ln\frac{2}{1-|z|^2} \leq ||W^n_{\psi,\varphi}||_{K_\alpha \rightarrow A^{\beta}_{ln}}.$

Taking supremum on both sides of above inequality, we have that (2.1) holds.

Theorem 2. Let $\alpha > 0, \beta > 0$, $n \in N \cup \{0\}$, $\psi \in H(\mathbb{D}), \varphi$ a holomorphic self-map of $\mathbb{D}$ and $d\lambda(z) = dA(z) / (1 - |z|^2)^2$. Then $W^n_{\psi,\varphi} : K_\alpha \rightarrow A^{\beta}_{ln}$ is bounded if and only if

$$L_1 = sup_{\zeta\in \boldsymbol{T}} sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|z|^2)^{2\alpha}}{|1-\bar{\zeta}\,\varphi(z)|^{2(n+\alpha)}} |\psi(z)|^2 ln\frac{2}{1-|z|^2} (1-|\gamma_a(z)|^2)^2 d\lambda(z) < \infty. \quad (2.6)$$

Proof: First assume that (2.6) holds. Let $D(a,(1-|a|)/2) = \{z \in \mathbb{D}: |z-a| < (1-|a|)/2\}$. Since

$(1-|a|^2)^{\alpha} ln\frac{2}{1-|a|^2} \asymp (1-|z|^2)^{\alpha} ln\frac{2}{1-|z|^2}$, for $z \in D(a,(1-|a|)/2)$. Using these two facts, (1.2) and the subharmonicity of the function

$g(z) = \frac{|\psi(z)|^2}{|1-\bar{\zeta}\,\varphi(z)|^{2(n+\alpha)}}$

we obtained

$$L_1 \geq sup_{\zeta\in \boldsymbol{T}} sup_{a\in\mathbb{D}} \int_{D(a,(1-|a|)/2)} \frac{|\psi(z)|^2}{|1-\bar{\zeta}\,\varphi(z)|^{2(n+\alpha)}} (1-|\gamma_a(z)|^2)^2 d\lambda(z)$$

$$= sup_{\zeta\in \boldsymbol{T}} sup_{a\in\mathbb{D}} \int_{D(a,(1-|a|)/2)} \frac{|\psi(z)|^2}{|1-\bar{\zeta}\,\varphi(z)|^{2(n+\alpha)}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z)$$

$$\geq sup_{\zeta\in \boldsymbol{T}} sup_{a\in\mathbb{D}} \frac{(1-|z|^2)^{2\alpha}}{|1-\bar{\zeta}\,\varphi(z)|^{2(n+\alpha)}} |\psi(z)|^2 ln\frac{2}{1-|z|^2} = M_1^2. \quad (2.7)$$

Thus by theorem 1, the operator $W^n_{\psi,\varphi} : K_\alpha \rightarrow A^{\beta}_{ln}$ is bounded.

Next assume that the operator $W^n_{\psi,\varphi} : K_\alpha \to A^\beta_{ln}$ is bounded. By theorem 1, we have that (2.1) holds. From this, we have

$$L_1 \le M_1^2 sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z) = M_1^2 C < \infty. \quad (2.8)$$

The asymptotic relation $L_1 \asymp M_1^2$ follows from (2.7) and (2.8).

Proceeding as in the proof of Theorem 2, we can easily prove the following lemma.

We omit the proof.

Lemma 1. Let$\alpha > 0, \beta > 0$ and $d\lambda(z) = dA(z) / (1 - |z|^2)^2$. Then $f \in A^\beta_{ln}$ if and only if

$$I := sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{2\alpha} ln \frac{2}{1-|z|^2} (1 - |\gamma_a(z)|^2)^2 d\lambda(z) < \infty.$$

Moreover, the following asymptotic relationship holds $||f||^2_{A^\beta_{ln}} \asymp I$.

By (1.3), the unit ball B_{K_α} of K_α is a normal family, a standard argument from Proposition 3.11 in [5] yields the proof of the next lemma.

Lemma 2. Let $\alpha > 0, \beta > 0, n \in N \cup \{0\}, \psi \in H(\mathbb{D}), \varphi$ a holomorphic self-map of $\mathbb{D}$**.** Then$W^n_{\psi,\varphi} : K_\alpha \to A^\beta_{ln}$ is compact if and only if any bounded sequence $\{f_m\}_{m \in \mathbb{N}}$ in K_α converging to zero on compacts subsets of $\mathbb{D}$, we have that $\lim_{m\to\infty} ||W^n_{\psi,\varphi} f_m||_{A^\beta_{ln}} = 0$.

Theorem 3.Let $\alpha > 0, \beta > 0, n \in N \cup \{0\}$, $\psi \in H(\mathbb{D}), \varphi$ a holomorphic self-map of $\mathbb{D}$ and $d\lambda(z) = dA(z) / (1 - |z|^2)^2$ and $W^n_{\psi,\varphi} : K_\alpha \to A^\beta_{ln}$ is bounded. Then the following statements are equivalent:

1. $W^n_{\psi,\varphi} : K_\alpha \to A^\beta_{ln}$is bounded.
2. $M_3 := sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (1 - |z|^2)^{2\alpha} ln \frac{2}{1-|z|^2} (1 - |\gamma_a(z)|^2)^2 |\psi(z)|^2 d\lambda(z) < \infty$ and

$$\lim_{r\to 1} sup_{\zeta \in \boldsymbol{T}} sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} \frac{(1-|z|^2)^{2\alpha}}{|1-\bar{\zeta}\varphi(z)|^{2(n+\alpha)}} ln \frac{2}{1-|z|^2} (1 - |\gamma_a(z)|^2)^2 |\psi(z)|^2 d\lambda(z) = 0. \quad (2.9)$$

Proof: (1) → (2). Since $W^n_{\psi,\varphi} : K_\alpha \to A^\beta_{ln}$ is bounded, for $(z) = z^n / n! \in K_\alpha$, we get

$$M_3 = sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (1 - |z|^2)^{2\alpha} ln \frac{2}{1-|z|^2} (1 - |\gamma_a(z)|^2)^2 |\psi(z)|^2 d\lambda(z) < \infty.$$

Let $f_m(z) = z^m, m \in \mathbb{N}$. It is norm bounded sequence in K_α converging to zero uniformly on compact subsets of D. Hence by Lemma 2, it follows that $||W^n_{\psi,\varphi} f_\zeta||_{A^\beta_{ln}} \to 0$ as $m \to \infty$. Thus for every $\epsilon > 0$, there is an $m_0 \in \mathbb{N}$ such that for $m \ge m_0$, we have

$$\left(\prod_{j=0}^n (m-j)\right)^2 sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi(z)|^{2(m-n)} (1 - |z|^2)^{2\alpha} ln \frac{2}{1-|z|^2} (1 - |\gamma_a(z)|^2)^2 |\psi(z)|^2 d\lambda(z) < \epsilon. \quad (2.10)$$

From (2.9), we have that for each $r \in (0,1)$

$$r^{2(m-n)} \left(\prod_{j=0}^n (m-j)\right)^2 sup_{a \in \mathbb{D}} \int_{|\varphi(z)|>r} (1 - |z|^2)^{2\alpha} ln \frac{2}{1-|z|^2} (1 - |\gamma_a(z)|^2)^2 |\psi(z)|^2 d\lambda(z) < \epsilon. \quad (2.11)$$

Hence for $\in [\prod_{j=0}^n (m-j)^{\frac{-1}{m-n}}, 1]$ we have

$sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}(1-|z|^2)^{2\alpha}\ln\frac{2}{1-|z|^2}(1-|\gamma_a(z)|^2)^2|\psi(z)|^2 d\lambda(z) < \epsilon.$ (2.12)

Let $f \in B_{K_\alpha}$ and $f_t(z) = f(tz),\ 0 < t < 1$. Then $sup_{0<t<1}||f_t||_{K_\alpha} \le ||f||_{K_\alpha}$, $f_t \in K_\alpha$, $t \in (0,1)$and $f_t \to f$ uniformly on compacts subset of $\mathbb{D}$ as $t \to 1$. The compactness of $W^n_{\psi,\varphi}: K_\alpha \to A^\beta_{ln}$ implies that

$\lim_{t\to1} ||W^n_{\psi,\varphi}f_t - W^n_{\psi,\varphi}f||_{A^\beta_{ln}} = 0$. Hence for every $\epsilon > 0$, there is a $t \in (0,1)$ such that

$sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|f^n_t(\varphi(z)) - f^n(\varphi(z))|^2(1-|z|^2)^{2\alpha}\ln\frac{2}{1-|z|^2}(1-|\gamma_a(z)|^2)^2|\psi(z)|^2 d\lambda(z) < \epsilon.$

(2.13)

By inequalities (2.12) and (2.13), we have

$sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|f^n(\varphi(z))|^2(1-|z|^2)^{2\alpha}\ln\frac{2}{1-|z|^2}(1-|\gamma_a(z)|^2)^2|\psi(z)|^2 d\lambda(z)$

$\le 2\, sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f^n_t(\varphi(z)) - f^n(\varphi(z))|^2(1-|z|^2)^{2\alpha}\ln\frac{2}{1-|z|^2}(1-|\gamma_a(z)|^2)^2|\psi(z)|^2 d\lambda(z)$

$+\ 2\, sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|f^n(\varphi(z))|^2(1-|z|^2)^{2\alpha}\ln\frac{2}{1-|z|^2}(1-|\gamma_a(z)|^2)^2|\psi(z)|^2 d\lambda(z)$

$\le 2\epsilon(1 + ||f^n_t||^2_\infty).$

Hence for every $f \in B_{K_\alpha}$, there is a $\delta_0 \in (0,1)$, $\delta_0 = \delta_0(f,\epsilon)$, such that for $r \in (\delta_0, 1)$

$sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|f^n(\varphi(z))|^2(1-|z|^2)^{2\alpha}\ln\frac{2}{1-|z|^2}(1-|\gamma_a(z)|^2)^2|\psi(z)|^2 d\lambda(z) < \epsilon.$

From the compactness of $W^n_{\psi,\varphi}: K_\alpha \to A^\beta_{ln}$, we have that for every $\epsilon > 0$ there is a finite collection of functions $f_1, f_2, f_3, \dots\dots, f_k \in B_{K_\alpha}$ such that for each $f \in B_{K_\alpha}$, there is a $j \in \{1,2,3,\dots..k\}$ such that

$sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f^n(\varphi(z)) - f^n_j(\varphi(z))|^2(1-|z|^2)^{2\alpha}\ln\frac{2}{1-|z|^2}(1-|\gamma_a(z)|^2)^2|\psi(z)|^2 d\lambda(z) < \epsilon.$ (2.14)

On the other hand, from (2.14) it follows that if $\delta := max_{1\le j\le k}\delta_j(f_j, \epsilon)$, then for $r \in (\delta, 1)$ and all $j \in \{1,2,3,\dots..k\}$ we have

$sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}f^n_j(\varphi(z)|)^2(1-|z|^2)^{2\alpha}\ln\frac{2}{1-|z|^2}(1-|\gamma_a(z)|^2)^2|\psi(z)|^2 d\lambda(z) < \epsilon.$ (2.15)

From (2.14) and (2.15), we have that for $r \in (\delta, 1)$ and every $f \in B_{K_\alpha}$

$sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}|f^n(\varphi(z))^2(1-|z|^2)^{2\alpha}\ln\frac{2}{1-|z|^2}(1-|\gamma_a(z))^2|^2|\psi(z)|^2 d\lambda(z) < 4\epsilon.$ (2.16)

Applying (2.16) to the functions $f_\zeta(z) = 1 / (1 - \bar{\zeta}z)^\alpha$, $\zeta \in \boldsymbol{T}$, we obtain

$sup_{\zeta\in\boldsymbol{T}} sup_{a\in\mathbb{D}}\int_{|\varphi(z)|>r}\frac{(1-|z|^2)^{2\alpha}}{|1-\bar{\zeta}\,\varphi(z)|^{2(n+\alpha)}}\ln\frac{2}{1-|z|^2}(1-|\gamma_a(z)|^2)^2|\psi(z)|^2 d\lambda(z) < 4\epsilon / (\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1))^2$ from which (2.9) follows.

(2) $\Rightarrow$ (1). Assume that $\{f_m\}_{m\in\mathbb{N}}$ is a bounded sequence inK_α, say by L, converging to 0 uniformly on compacts of $\mathbb{D}$ as $m \to \infty$. Then by the Weierstrass's theorem, $f^{(k)}_m$ also converges to 0 uniformly on compacts of $\mathbb{D}$, for each $k \in \mathbb{N}$. We need to show that $||W^n_{\psi,\varphi}f_m||_{A^\beta_{ln}} \to 0$ as $m \to \infty$. For each $m \in \mathbb{N}$, we can find a $\mu_m \in \mathcal{M}$ with $||\mu_m||$ = $||\mu_m||_{K_\alpha}$ such that

$f_m(z)= \int \boldsymbol{T} \frac{1}{(1-\bar{\zeta}z)^{\alpha}} d\mu_m(\zeta)$ (2.17)

Differentiating (2.17) n times, composing such obtained equation by φ, applying Jensen's inequality,as well as the boundedness of sequence$\{f_m\}_{m\in\mathbb{N}}$, we obtain

$|f_m^{(n)}(\varphi(w))|^2 \leq L(\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1))^2 \int \boldsymbol{T} \frac{1}{|1-\bar{\zeta}\varphi(w)|^{2(n+\alpha)}} d|\mu_m|(\zeta).$ (2.18)

By the second condition in (2), we have that for every$\epsilon > 0$ there is an $r_1 \in (0,1)$ such that for$r \in (r_1, 1)$, we have

$sup_{\zeta\in\boldsymbol{T}} sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{(1-|z|^2)^{2\alpha}}{|1-\bar{\zeta}\varphi(z)|^{2(n+\alpha)}} ln \frac{2}{1-|z|^2} (1-|\gamma_a(z)|^2)^2 |\psi(z)|^2 d\lambda(z) < \epsilon.$ (2.19)

By (1.3), we have

$||W_{\psi,\varphi}^n f_m||_{A_{ln}^{\beta}} \asymp sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\leq r} |f_m^{(n)}(\varphi(w)|))^2$ (1-
$(z)|^2)^2 |\psi(z)|^2 (1-|z|^2)^{2\alpha} ln \frac{2}{1-|z|^2} d\lambda(z) + sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f_m^{(n)}(\varphi(w)|))^2 (1-$
$|\gamma_a(z)|^2)^2 |\psi(z)|^2 (1-|z|^2)^{2\alpha} ln \frac{2}{1-|z|^2} d\lambda(z).$

Using first condition in (2), (2.19), Fubini's theorem and the fact that $sup_{|w|\leq r} |f_m^{(n)}(w)|^2 < \epsilon$, for sufficiently large m,say$m \geq m_0$, we have that

$||W_{\psi,\varphi}^n f_m||_{A_{ln}^{\beta}} \leq sup_{\varphi(z)\leq r} |f_m^{(n)}(\varphi(w))|^2 sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\leq r} (1-$
$|\gamma_a(z)|^2)^2 |\psi(z)|^2 (1-|z|^2)^{2\alpha} ln \frac{2}{1-|z|^2} d\lambda(z) + sup_{a\in\mathbb{D}} \int_{\boldsymbol{T}} \int_{|\varphi(z)|>r} \frac{(1-|z|^2)^{2\alpha}}{|1-\bar{\zeta}\varphi(z)|^{2(n+\alpha)}} ln \frac{2}{1-|z|^2} (1-$
$|\gamma_a(z)|^2)^2 |\psi(z)|^2 d\lambda(z) d|\mu_m|(\zeta) \leq (M_3 + \int_{\boldsymbol{T}} d|\mu_m|(\zeta)) \leq (M_3 + L)\,\epsilon.$

Since ϵ is an arbitrary, the result follows by Lemma2.

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References

[1] Bourdon, P., and Cima, J. A., On integrals of Cauchy-Stieltjes type. Houston J. Math. **14**, 465-474(1988).

[2] Choa, J. S., and Kim, H. O., Composition operators from the space of Cauchy transforms into its Hardy type subspaces. Rocky Mountain J. Math., **31**, 95-113(2001).

[3] Cima, J. A., and Matheson, A. L., Cauchy transforms and composition operators. Illinois J. Math.**4**, 58-69(1998).

[4] Cima, J. A., and MacGregor, T. H., (1987). Cauchy transforms of measures and univalent functions. Lecture Notes in Math. 1275 (pp. 78- 88). Springer- Verlag.

[5] Cowen, C. C., and MacCluer, (1995).Composition operators on spaces of analytic functions, CRC Press Boca Raton, New York.

[6] Hibschweiler, R. A., Composition operators on spaces of Cauchy transforms. Contemp. Math. **213**, 57-63 (1998).

[7] Hibschweiler, R. A., Composition operators on spaces of fractional Cauchy transforms.Complex Anal. Oper. Theory **6**, 897-911(2012).

[8] Hibschweiler, R. A., and MacGregor, T. H., Closure properties of families of Cauchy- Stieltjes transforms. Proc. Amer. Math. Soc. **105**, 615-621 (1989).

[9] Hibschweiler, R. A., and MacGregor, T. H., Multipliers of families of Cauchy- Stieltjes transforms. Trans. Amer. Math. Soc. **331**, 377-394 (1992).

[10] Hibschweiler, R. A., and MacGregor, T. H., Bounded analytic families of Cauchy-Stieltjes integrals. Rocky Mountain J. Math. **23**, 187-202 (1993).

[11] Sharma, A. K., and Sharma, A., Integration operators from space of Cauchy integral transforms to the Dirichlet space. Adv. Pure Appl. Math. **5**, 47-53 (2014).

[12] Sharma, Ajay K., Weighted composition operators from Cauchy Integral transforms to logarithmic weighted- type spaces. Ann. Funct. Anal. **4**, 163-174 (2013).

[13] Stevic, S., and Sharma, A. K., Composition operators from the space of Cauchy transforms to Bloch and the little Bloch-type spaces on the unit disk. Appl. Math. Comput. **217**, 10187-10194 (2011).