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A note on bicomplex Orlicz spaces

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Abstract

The set $\mathbb{BC}$ of bicomplex numbers is defined as $\mathbb{BC} = \{z_1 + z_2 j : z_1, z_2 \in \mathbb{C}(i)\}$, where i and j are independent imaginary units such that $i^2 = j^2 = -1$ and $\mathbb{C}(i) = \{x + iy : x, y \in \mathbb{R}\}$. In this paper we studied bicomplex function spaces and in particular we studied bicomplex Orlicz spaces.

1 INTRODUCTION

In this section we summarize some basic properties already established for the bicomplex numbers and also give some remarks and conclusions. The set $\mathbb{BC}$ of bicomplex numbers is defined as

$$\mathbb{BC} = \{z_1 + z_2 j : z_1, z_2 \in \mathbb{C}(i)\},$$

where i and j are independent imaginary units such that $i^2 = j^2 = -1$ and $\mathbb{C}(i) = \{x+iy : x, y \in \mathbb{R}\}$. The product of any two units is commutative and satisfies

$$ij = ji = k, ik = -j, jk = -i.$$

The set of bicomplex numbers, $\mathbb{BC}$, can also be defined as

$$\mathbb{BC} = \{a + bi + cj + dk : a, b, c, d, \in \mathbb{R}\}.$$

Under addition and multiplication defined as :

$$\begin{aligned} Z + W &= (z_1 + z_2 j) + (w_1 + w_2 j) \\ &= (z_1 + w_1) + (z_2 + w_2) j \\ Z.W &= (z_1 + z_2 j).(w_1 + w_2 j) \\ &= (z_1 w_1 - z_2 w_2) + (z_1 w_2 + z_2 w_1) j \end{aligned}$$

$\mathbb{BC}$ becomes a commutative ring with unity and therefore a module over itself. Also since the field $\mathbb{C}(i)$ is a subring of $\mathbb{BC}$, therefore $\mathbb{BC}$ can also be seen as a vector space over $\mathbb{C}(i)$. If we put $z_1 = x$ and $z_2 = iy$ with $x, y \in \mathbb{R}$, then we obtain the set of hyperbolic numbers

$$\mathbb{D} = \{x + yk : k^2 = 1, x, y \in \mathbb{R}\}.$$

Three types of conjugations can be defined on the set of bicomplex numbers. With $Z = z_1 + z_2 j$, we define

(i) $\overline{Z} = \overline{z}_1 + \overline{z}_2 j$

(ii) $Z^\dagger = z_1 - z_2 j$

(iii) $Z^* = (\overline{Z})^* = \overline{z}_1 - \overline{z}_2 j.$

Each conjugate is additive, multiplicative and involute on $\mathbb{BC}$. Let $e = \dfrac{1+ij}{2}$. Then the $\dagger$ - conjugate of e is given by $e^\dagger = \dfrac{1-ij}{2}$. Any bicomplex number $Z = z_1 + z_2 j$ can be uniquely expressed as $Z = \beta_1 e + \beta_2 e^\dagger$, where $\beta_1 = z_1 - z_2 i$ and $\beta_2 = z_1 + z_2 i$ are in $\mathbb{C}(i)$ The bicomplex numbers e and $e^\dagger$ are hyperbolic numbers with

$$\begin{array}{lll} e^2 = e; & e^* = e; & e + e^\dagger = 1; \\ (e^\dagger)^2 = e^\dagger; & (e^\dagger)^* = e^\dagger; & e.e^\dagger = 0. \end{array}$$

e and $e^\dagger$ form the idempotent basis of bicomplex numbers. The uniqueness of idempotent representation of bicomplex numbers allows the introduction of two projection operators $\pi_{1,i} : \mathbb{BC} \mapsto \mathbb{C}(i)$ and $\pi_{2,i} : \mathbb{BC} \mapsto \mathbb{C}(i)$ defined as

$$\pi_{l,i}(Z) = \pi_{l,i}(\beta_1 e + \beta_2 e^\dagger) = \beta_l \in \mathbb{C}(i) \text{ for } l = 1, 2,$$

with the property that $\pi_{1,i}e + \pi_{2,i}e^\dagger = I$ and both the projection maps are additive and multiplicative, i.e. $\pi_{l,i}(Z + W) = \pi_{l,i}(Z) + \pi_{l,i}(W)$ and $\pi_{l,i}(Z.W) = \pi_{l,i}(Z).\pi_{l,i}(W)$ for all $Z, W \in \mathbb{BC}$ and $l = 1, 2$.

For $Z = z_1 e + z_2 e^\dagger \in \mathbb{BC}$, the norm of Z is defined as $\|Z\| = \dfrac{1}{\sqrt{2}}\sqrt{|z_1|^2 + |z_2|^2}$.

Let $A \subseteq \mathbb{BC}$. Also each $x \in A$ is of the form $Z = z_1 e + z_2 e^\dagger$ where z_1 and z_2 are in $\mathbb{C}(i)$. Corresponding to each $A \subseteq \mathbb{BC}$, we associate two subsets of $\mathbb{C}(i)$, viz. A_e and $A_{e^\dagger}$ comprising of $\pi_{l,i}(Z)$ and $\pi_{2,i}(Z)$, respectively, for all $Z \in A$. Therefore $A_e = \pi_{1,i}(A) = eA$ and $A_{e^\dagger} = \pi_{2,i}(A) = e^\dagger A$.

Also let A_1 and A_2 be two sets defined as

$$\begin{aligned} A_1 &= \{z_1 - iz_2 : z_1, z_2 \in \mathbb{C}(i)\} \\ A_2 &= \{z_1 - iz_2 : z_1, z_2 \in \mathbb{C}(i)\}. \end{aligned}$$

Then A_1 and A_2 are copies of $\mathbb{C}(i)$ such that $\mathbb{BC} = A_1 e + A_2 e^\dagger$. Therefore for any subset U of $\mathbb{BC}$, we have $U = U_1 e + U_2 e^\dagger$, where U_1 and U_2 are subsets of A_1 and A_2 respectively. It is easy to show now that the characteristic functions χ_U is given by $\chi_U = \chi_{U_1}\chi_{U_2}e + \chi_{U_1}\chi_{U_2}e^\dagger$.

Remark 1.1. $\mathbb{BC}$ together with the norm defined above form a generalized normed algebra, since $\|Z.W\| \leq \frac{1}{\sqrt{2}}\|Z\|.\|W\|$. Also since $\mathbb{BC} \simeq \mathbb{R}^4$ and $\mathbb{R}^4$ is complete with respect to usual metric, it follows that $\mathbb{BC}$ forms a generalized Banach algebra. The product of two bicomplex numbers $Z = \beta_{Z,1}e + \beta_{Z,2}e^\dagger$ and $W = \beta_{W,1}e + \beta_{W,2}e^\dagger$ can be written in the idempotent basis as $Z.W = (\beta_{Z,1}e + \beta_{Z,2}e^\dagger).(\beta_{W,1}e + \beta_{W,2}e^\dagger) = \beta_{Z,1}.\beta_{W,1}e + \beta_{Z,2}.\beta_{W,2}e^\dagger$. Thus we see that Z is invertible if and only if $\beta_{Z,1} \neq 0 \neq \beta_{Z,2}$ and $Z^{-1} = \beta_{Z,1}^{-1}e + \beta_{Z,2}^{-1}e^\dagger$. A non zero Z that does not have an inverse has the property that either $\beta_{Z,1} = 0$ or $\beta_{Z,2} = 0$ and such a Z is divisor of zero. The zero divisors make up the so called null cone NC which is closed subset of $\mathbb{BC}$. Therefore the set of invertible elements form an open subset of $\mathbb{BC}$ and every zero divisor is the limit point of the set of regular (invertible) elements in $\mathbb{BC}$. The existence of zero divisors in $\mathbb{BC}$ and the solution of polynomial equation emphasizes the following difference. It comes out that the equation $\sum_{k=1}^{n}(a_k + jb_k)(Z = z_1 + z_2 j)^k = 0, a_n^2 + b_n^2 \neq 0$ has n^2 solutions in $\mathbb{BC}$ whereas such type of equation in $\mathbb{C}$ has n solutions.

In [8], Hahn Banach Theorem, Closed Graph Theorem, Open Mapping Theorem and Uniform Boundedness Principle for bicomplex Banach spaces were established. For various properties of finite and infinite dimensional bicomplex Hilbert spaces and their applications one can refer to [6], [15] and [16]. Recently D. Alpay, M. E. Luna - Elizarraras, M. Shapiro and D. C. Struppa have written a clear and nice paper on bicomplex function analysis, see [1].

2 Bicomplex function spaces and linear operators

In this section we define bicomplex valued function spaces and give some examples of linear operators.

Let $\Omega = (\Omega, \Sigma, \mu)$ be a σ-finite complete measure space. If $f = f_1 e + f_2 e^{\dagger}$, where f_1 and f_2 are complex ($\mathbb{C}(i)$) valued measurable functions on $\Omega = (\Omega, \Sigma, \mu)$, then f is a bicomplex valued measurable function on Ω. Therefore given any complex ($\mathbb{C}(i)$) valued function space $(F(\Omega), \|.\|_{\Omega})$ we can always define a bicomplex version $(F(\Omega, \mathbb{BC}), \|.\|_{\mathbb{BC}})$ comprising of all functions of the type $f = f_1 e + f_2 e^{\dagger}$, where f_1 and f_2 are in $(F(\Omega), \|.\|_{\Omega})$ and $\|f\|_{\mathbb{BC}} = \frac{1}{\sqrt{2}}(\|f_1\|_{\Omega}^2 + \|f_2\|_{\Omega}^2)^{\frac{1}{2}}$. The addition and scalar multiplication is defined on $(F(\Omega, \mathbb{BC}), \|.\|_{\mathbb{BC}})$ as under:

$$
\begin{aligned}
f + g &= (f_1 e + f_2 e^{\dagger}) + (g_1 e + g_2 e^{\dagger}) \\
&= (f_1 + g_1)e + (f_2 + g_2)e^{\dagger}
\end{aligned}
$$

and

$$
\begin{aligned}
a.f &= (a_1 e + a_2 e^{\dagger}).(f_1 e + f_2 e^{\dagger}) \\
&= (a_1.f_1)e + (a_2.f_2)e^{\dagger},
\end{aligned}
$$

where $f, g \in F(\Omega, \mathbb{BC})$ and $a \in \mathbb{BC}$. One can easily see that the bicomplex version of complex valued function space is complete if and only if the complex valued function space itself is complete.

Example 2.1. Suppose $L^0(\Omega)$ denotes the linear space of all equivalence classes of complex valued Σ-measurable functions on Ω and here any two functions that are equal μ-almost everywhere on Ω are identified. Then the corresponding bicomplex measurable function space $L^0(\mathbb{BC})$ comprises of all functions of the type $f = f_1 e + f_2 e^{\dagger}$ where $f_1, f_2 \in L^0(\Omega)$.

Example 2.2. Suppose $L^{\infty}(\Omega)$ denotes the linear space of all equivalence classes of complex valued Σ-measurable essentially bounded functions on Ω and here any two functions that are equal μ-almost everywhere on Ω are identified.Then the corresponding bicomplex measurable function space $L^{\infty}(\mathbb{BC})$ comprises of all functions of the type $f = f_1 e + f_2 e^{\dagger}$ where $f_1, f_2 \in L^{\infty}(\Omega)$. Also $\|f\|_{\infty,\mathbb{BC}} = \|f_1 e + f_2 e^{\dagger}\|_{\infty,\mathbb{BC}} = \frac{1}{\sqrt{2}}(\|f_1\|_{\infty}^2 + \|f_2\|_{\infty}^2)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}((esssup f_1)^2 + (esssup f_2)^2)^{\frac{1}{2}}$

Similarly we can define $L^p(\mathbb{BC})$.

Lemma 2.3. *$(F(\Omega, \mathbb{BC}), \|.\|_{\mathbb{BC}})$ is complete if and only if $(F(\Omega), \|.\|_{\Omega})$ is complete.*

Proof. Suppose $(F(\Omega, \mathbb{BC}), \|.\|_{\mathbb{BC}})$ is complete and $\{f_n\}$ is a Cauchy sequence in $(F(\Omega), \|.\|_{\Omega})$. Therefore for given $\epsilon > 0$ there exists $r \in \mathbb{N}$ such that

$$\|f_n - f_m\|_{\Omega} < \epsilon \tag{2.1}$$

for all $n, m > r$. Set $f_n^{'} = f_n e + 0e^{\dagger} \in \mathbb{BC}$.Then

$$\begin{aligned}\|f_n^{'} - f_m^{'}\|_{\mathbb{BC}}^2 &= \|(f_n e + 0e^{\dagger}) - (f_m e + 0e^{\dagger})\|_{\mathbb{BC}}^2 \\ &= \|(f_n - f_m)e + 0e^{\dagger}\|_{\mathbb{BC}}^2 \\ &= \frac{1}{2}(\|f_n - f_m\|_{\Omega}^2).\end{aligned}$$

This together with (2.1) implies that $\{f_n^{'}\}$ is a Cauchy in $(F(\Omega, \mathbb{BC}), \|.\|_{\mathbb{BC}})$. Therefore by completeness of $(F(\Omega, \mathbb{BC}), \|.\|_{\mathbb{BC}})$, there exists $g = g_1 e + g_2 e^{\dagger}$ in $F(\Omega, \mathbb{BC})$ such that $f_n^{'} \to g$ as $n \to \infty$. In order to complete the proof of direct part we need to show that

$f_n \to g_1$ as $n \to \infty$ and $g_2 = 0$. Now $f_n' \to g$ in $\mathbb{BC}$, therefore there exist a natural number k such that

$$\|f_n' - g\|_{\mathbb{BC}} < \epsilon \tag{2.2}$$

for all $n > k$. Also one has

$$\begin{aligned}
\|f_n' - g\|_{\mathbb{BC}}^2 &= \|(f_n e + 0e^\dagger) - (g_1 e + g_2 e^\dagger)\|_{\mathbb{BC}}^2 \\
&= \|(f_n - g_1)e + (0 - g_2)e^\dagger\|_{\mathbb{BC}}^2 \\
&= \frac{1}{2}(\|f_n - g_1\|_\Omega^2 + \|0 - g_2\|_\Omega^2).
\end{aligned}$$

This together with (2.2) implies the direct part. For converse, suppose $(F(\Omega), \|.\|_\Omega)$ is complete and $\{f_n = f_{n,1}e + f_{n,2}e^\dagger\}_{n=1}^\infty$ is a Cauchy sequence in $(F(\Omega, \mathbb{BC}), \|.\|_{\mathbb{BC}})$. Therefore for $\epsilon > 0$ there exists $r \in \mathbb{N}$ such that

$$\|f_m - f_n\|_{\mathbb{BC}} < \epsilon \tag{2.3}$$

for all $m, n > r$. But

$$\begin{aligned}
\|f_m - f_n\|_{\mathbb{BC}}^2 &= \|(f_{m,1}e + f_{m,2}e^\dagger) - (f_{n,1}e + f_{n,2}e^\dagger)\|_{\mathbb{BC}}^2 \\
&= \|(f_{m,1} - f_{n,1})e + (f_{m,2} - f_{n,2})e^\dagger\|_{\mathbb{BC}}^2 \\
&= \frac{1}{2}(\|f_{m,1} - f_{n,1}\|_\Omega^2 + \|f_{m,2} - f_{n,2}\|_\Omega^2)
\end{aligned}$$

Therefore by (2.3) we have $\|f_{m,i} - f_{n,i}\|_\Omega < \sqrt{2}\epsilon$ for $i = 1, 2$ and for all $m, n > r$. This implies that $\{f_{n,i}\}$ is a Cauchy sequence in $(F(\Omega), \|.\|_\Omega)$. Since $(F(\Omega), \|.\|_\Omega)$ is complete, therefore there exists f_i in $(F(\Omega), \|.\|_\Omega)$ such that $f_{n,i} \to f_i$ as $n \to \infty$ for $i = 1, 2$. Therefore there exist natural numbers $k_1 > 0$ and $k_2 > 0$ such that

$$\|f_{n,i} - f_i\|_\Omega < \epsilon \tag{2.4}$$

for all $n > k_i$. Next we claim that $f_n = f_{n,1}e + f_{n,2}e^\dagger \to f = f_1 e + f_2 e^\dagger$ as $n \to \infty$. Now

$$\begin{aligned}\|f_n - f\|_{\mathbb{BC}}^2 &= \|(f_{n,1}e + f_{n,2}e^\dagger) - (f_1 e + f_2 e^\dagger)\|_{\mathbb{BC}}^2 \\ &\|(f_{n,1} - f_1)e + (f_{n,2} - f_2)e^\dagger\|_{\mathbb{BC}}^2 \\ &= \frac{1}{2}(\|f_{n,1} - f_1\|_\Omega^2 + \|f_{n,2} - f_2\|_\Omega^2) \\ &< \epsilon^2\end{aligned}$$

for all $n > k = max\{k_1, k_2\}$. Therefore one has $\|f_n - f\|_{\mathbb{BC}} < \epsilon$ for all $n > k$. Therefore f_n is convergent in $(F(\Omega, \mathbb{BC}), \|.\|_{\mathbb{BC}})$. ■

Definition 2.4. A bicomplex linear operator T is a mapping from a bicomplex normed linear space X to X such that $T(\alpha z + \beta w) = \alpha T(z) + \beta T(w)$, $\forall \alpha, \beta \in \mathbb{BC}$ and z,w $\in$ X.

Remark 2.5. A bicomplex linear operator $T : X \mapsto X$ can always be written as T $= eT_1 + e^\dagger T_2 = e\pi_{i,1}(T) + e^\dagger \pi_{i,2}(T)$, where $\pi_{i,l}(T)(l = 1, 2)$ is defined as $\pi_{i,l}(T) = T_l$.

Proposition 2.6. *A linear operator $T : X \mapsto X$ is invertible if and only if both T_1 and T_2 are invertible and $T^{-1} = eT_1^{-1} + e^\dagger T_2^{-1}$.*

Proof. For proof see [3] ■

Remark 2.7. Let $T = T_1 e + T_2 e^\dagger$ be a bicomplex linear operator on $(F(\Omega, \mathbb{BC}), \|.\|_{\mathbb{BC}})$.

(a) A bicomplex linear operator T is bounded on a bicomplex function space $(F(\Omega, \mathbb{BC}), \|.\|_{\mathbb{BC}})$ if and only if both T_1 and T_2 are bounded on the underlying complex valued function space $(F(\Omega), \|.\|_\Omega)$, see [8].

(b) A bicomplex linear operator T is compact on a bicomplex function space $(F(\Omega, \mathbb{BC}), \|.\|_{\mathbb{BC}})$ if and only if both T_1 and T_2 are compact on the underlying complex valued function space $(F(\Omega), \|.\|_\Omega)$, see [2].

Next we define a bicomplex valued Orlicz space. An Orlicz function $\phi : [0, \infty) \mapsto [0, \infty]$ is a convex function with $\phi(0) = 0$ and $\phi(u) \mapsto \infty$ as $u \mapsto \infty$ such that $\phi(u) < \infty$ for some $0 < u < \infty$. An Orlicz space $L^\phi(\Omega)$ is defined as the space of all $f \in L^\circ(\Omega)$ such that $I_\phi(|\lambda f|)$ is finite for some $\lambda > 0$ and for any $f \in L^\phi(\Omega)$, the Orlicz norm of f, $\|f\|_\phi$ is defined as the infimum of all $\lambda > 0$ such that $I_\phi(\frac{f}{\lambda}) \leq 1$ where, $I_\phi(f) = \int_\Omega \phi(|f|)\, d\mu$. The Orlicz space $L^\phi(\Omega)$ is a Banach space with Orlicz norm $\|.\|_\phi$.

The bicomplex Orlicz space $L^\phi(\mathbb{BC})$ is defined as

$$L^\phi(\mathbb{BC}) = \{f_1 e + f_2 e^\dagger : f_1, f_2 \in L^\phi(\mu)\}.$$

Also the fact that the norm $\|.\|_{\phi,\mathbb{BC}} : L^\phi(\mathbb{BC}) \mapsto \mathbb{R}$ defined as

$$\|f\|_{\phi,\mathbb{BC}} = \frac{1}{\sqrt{2}}(\|f_1\|_\phi^2 + \|f_2\|_\phi^2)^{\frac{1}{2}}$$

is a complete norm on $L^\phi(\mathbb{BC})$ follows easily from Lemma 2.3.

Remark 2.8. If $\phi(x) = |x|^P$ for $p > 1$, we obtain the bicomplex version of the Lebesgue spaces L^p.

Now we give some examples of linear operators:

Definition 2.9. Let $T : \Omega \mapsto \Omega$ be a measurable transformation, that is, $T^{-1}(A) \in \Sigma$ for any $A \in \Sigma$. If $\mu o T^{-1}(A) = 0$ for each $A \in \Sigma$ with $\mu(A) = 0$, then T is said to be non-singular.

Any non-singular measurable transformation T induces a linear operator C_T from $L^0(\Omega)$ into itself defined by

$$(C_T f)(t) = f o T(t) = f(T(t)),\ \ t \in \Omega,\ \ f \in L^0(\Omega).$$

Next if $f = f_1 e + f_2 e^\dagger \in L^0(\mathbb{BC})$, then one can define

$$(C_T f)(t) = f_1 o T(t) e + f_2 o T(t) e^\dagger$$

Definition 2.10. Let $\Omega = (\Omega, \Sigma, \mu)$ be a σ-finite complete measure space and let $f = f_1 e + f_2 e^\dagger \in L^\phi(\mathbb{BC})$, where f_1 and f_2 are in L^ϕ. For a nonsingular measurable transformation $T : \Omega \mapsto \Omega$ define composition transformation $C_T f = C_T(f_1 e + f_2 e^\dagger) = f_1 \circ T e + f_2 \circ T e^\dagger$. We say C_T is composition operator if it maps $L^\phi(\mathbb{BC})$ into itself.

Remark 2.11. Clearly one has $C_T(f_1 e + f_2 e^\dagger) = C_T f_1 e + C_T f_2 e^\dagger$.

Remark 2.12. In light of Remark 2.7, [4] and [17], it easily follows that $C_T : L^p(\mathbb{BC}) \mapsto L^p(\mathbb{BC})$ is bounded if and only if $C_T : L^p(\Omega) \mapsto L^p(\Omega)$ is bounded i.e., if and only if

$$\mu o T^{-1}(A) \leq M \mu(A),$$

for some $M > 0$ and for all $A \in \Sigma$ with $\mu(A) < \infty$.

Example 2.13. Let $T : \mathbb{N} \mapsto \mathbb{N}$ be defined by $T(n) = n-1$. Then $C_T : l^2(\mathbb{BC}) \mapsto l^2(\mathbb{BC})$ is a composition operator on $l^2(\mathbb{BC})$ defined by $C_T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ and is called the right shift operator.

Definition 2.14. Let $\theta \in L^0(\mathbb{BC})$. Then we define a multiplication operator $M_\theta : L^0(\mathbb{BC}) \mapsto L^0(\mathbb{BC})$ as

$$M_\theta f(t) = (\theta(t))(f(t)),$$

for all $t \in \Omega$ and $f \in L^0(\mathbb{BC})$.

Remark 2.15. Clearly $\theta, f \in L^0(\mathbb{BC})$ implies that $\theta = \theta_1 e + \theta_2 e^\dagger$ for some $\theta_1, \theta_2 \in L^0(\Omega)$ and $f = f_1 e + f_2 e^\dagger$ for some $f_1, f_2 \in . \in L^0(\Omega)$ Therefore

$$\begin{aligned} M_\theta &= (\theta)(f) \\ &= (\theta_1 e + \theta_2 e^\dagger)(f_1 e + f_2 e^\dagger) \\ &= (\theta_1)(f_1)e + (\theta_2)(f_2 e^\dagger) \\ &= (M_{\theta_1} f_1)e + (M_{\theta_2} f_2)e^\dagger \end{aligned}$$

where M_{θ_1} and M_{θ_2} are multiplication operators on $L^0(\Omega)$. Therefore M_θ is bounded on $L^0(\mathbb{BC})$ if and only if M_{θ_1} and M_{θ_2} are bounded on $L^0(\Omega)$ i.e., if and only if $\theta_1, \theta_2 \in L^\infty(\Omega)$.

Lemma 2.16. *Let $1 < p < \infty$. Then the sequence of unit vectors $\{e_1, e_2, e_3, \ldots\}$, where $e_k = \delta_{kj}$ is a Schauder basis for $l^p(\mathbb{BC})$.*

Proof. For any $x = \{x_n\} \in l^p(\mathbb{BC})$,

$$\begin{aligned} \|x - \sum_{k=1}^{n} x_k e_k\| &= \|\{x_{n+1}, x_{n+2}, x_{n+3}, \ldots\}\| \\ &= (\sum_{k=n+1}^{\infty} |x_k|^p)^{\frac{1}{p}} \\ &= (\sum_{k=n+1}^{\infty} |e x_{k1} + e^\dagger x_{k2}|^p)^{\frac{1}{p}} \\ &\leq (\sum_{k=n+1}^{\infty} |e x_{k1}|^p + \sum_{k=n+1}^{\infty} |e^\dagger x_{k2}|^p)^{\frac{1}{p}} \\ &= (\sum_{k=n+1}^{\infty} |e|^p |x_{k1}|^p + \sum_{k=n+1}^{\infty} |e^\dagger|^p |x_{k2}|^p)^{\frac{1}{p}} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore $x = \sum_{k=1}^{\infty} x_k e_k$ and hence $\{e_1, e_2, e_3, \ldots\}$ is a Schauder basis for $l^p(\mathbb{BC})$. ■

Theorem 2.17. *The dual space of $l^p(\mathbb{BC})$ is $l^q(\mathbb{BC})$ where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Since by Lemma 2.16, the sequence of unit vectors $\{e_1, e_2, e_3, \ldots\}$, where $e_k = \delta_{kj}$ is a Schauder basis for $l^p(\mathbb{BC})$. The rest of the proof follows on the similar lines as in the case of $l^p(\Omega)$. ■

Next we define modular $I_{\phi,\mathbb{BC}}$ on bicomplex Orlicz space as

$$I_{\phi,\mathbb{BC}}(f) = I_\phi(f_1)e + I_\phi(f_2)e^\dagger \tag{2.5}$$

Theorem 2.18. *Let $f = f_1 e + f_2 e^\dagger \in L^\phi(\mathbb{BC})$, where f_1 and f_2 are in $L^\phi(\mu)$.*

(a) If $\|f\|_{\phi,\mathbb{BC}} \le \frac{1}{\sqrt{2}}$ then $I_\phi(f_i) \le 1$, for $i = 1, 2$.

(b) If $I_\phi(f_i) \le 1$, for $i = 1, 2$ then $\|f\|_{\phi,\mathbb{BC}} \le 1$.

Proof.

(a) Let $\|f\|_{\phi,\mathbb{BC}} = \frac{1}{\sqrt{2}}(\|f_1\|_\phi^2 + \|f_2\|_\phi^2)^{\frac{1}{2}} \le \frac{1}{\sqrt{2}}$. This implies that $\|f\|_{\phi,\mathbb{BC}}^2 = \frac{1}{2}(\|f_1\|_\phi^2 + \|f_2\|_\phi^2) \le \frac{1}{2}$. Which yields that $\|f_i\|_\phi^2 \le 1$, for $i = 1, 2$. Therefore $I_\phi(f_i) \le 1$, for $i = 1, 2$.

(b) Let $I_\phi(f_i) \le 1$, for $i = 1, 2$. This yields that $\|f_i\|_\phi \le 1$, for $i = 1, 2$. Therefore $\|f\|_{\phi,\mathbb{BC}} = \frac{1}{\sqrt{2}}(\|f_1\|_\phi^2 + \|f_2\|_\phi^2)^{\frac{1}{2}} \le 1$. ■

A particular case of $L^\phi(\mathbb{BC})$ is $l^\phi(\mathbb{BC})$ which is defined as

$$l^\phi(\mathbb{BC}) = \{\{x_n\}_{n=1}^{\infty} : x_n = x_{n,1}e + x_{n,2}e^\dagger \in \mathbb{BC} \text{ and } \{x_{n,i}\}_{n=1}^{\infty} \in l^\phi(\mathbb{C}) \text{ for } i = 1, 2.\}$$

and $\|\{x_n\}\|_{\phi,\mathbb{BC}} = \frac{1}{\sqrt{2}}(\|\{x_{n,1}\}\|_\phi^2 + \|\{x_{n,2}\}\|_\phi^2)^{\frac{1}{2}}$. The addition and scalar multiplication is defined component wise on $l^\phi(\mathbb{BC})$.

Theorem 2.19. *$l^\phi(\mathbb{BC})$ equipped with $\|.\|_{\phi,\mathbb{BC}}$ is a Banach space over $\mathbb{BC}$.*

Proof. It is clear from the definition of addition and scalar multiplication that $l^\phi(\mathbb{BC})$ is a module over $\mathbb{BC}$. It is also easy to show that $\|.\|_{\phi,\mathbb{BC}}$ defines a norm on $l^\phi(\mathbb{BC})$. Now it only remains to show that $l^\phi(\mathbb{BC})$ is also complete with respect $\|.\|_{\phi,\mathbb{BC}}$. For that let $\{x_n^k\}_{k=1}^\infty$ be a Cauchy sequence in $l^\phi(\mathbb{BC})$. Since for $\{x_n = x_{n,1}e + x_{n,2}e^\dagger\}_{n=1}^\infty$ we have $\|\{x_n\}\|_{\phi,\mathbb{BC}} = \frac{1}{\sqrt{2}}(\|\{x_{n,1}\}\|_\phi^2 + \|\{x_{n,2}\}\|_\phi^2)^{\frac{1}{2}}$. Therefore one has

$$\|\{x_{n,i}\}\|_\phi \leq \sqrt{2}\|\{x_n\}\|_{\phi,\mathbb{BC}} \text{ for } i = 1, 2. \tag{2.6}$$

Now for $\epsilon > 0$ there exists $r \in \mathbb{N}$ Such that

$$\|\{x_n^l - x_n^m\}\|_{\phi,\mathbb{BC}} < \frac{\epsilon}{\sqrt{2}} \text{ for all } l, m \geq r. \tag{2.7}$$

Therefore (2.6) and (2.7) yield that $\{x_{n,i}^k\}_{k=1}^\infty$ is a Cauchy sequence in $(l^\phi(\mathbb{C}), \|.\|_\phi)$ for $i = 1, 2$. In view of the completeness of the $(l^\phi(\mathbb{C}), \|.\|_\phi)$, we have that $\{x_{n,i}^k\}_{k=1}^\infty$ converges to some $\{x_{n,i}\} \in l^\phi(\mathbb{C})$. Therefore for $i = 1, 2$ there exists $p_i \in \mathbb{N}$ such that

$$\|\{x_{n,i}^l - x_{n,i}\}\|_\phi < \sqrt{\epsilon} \text{ for all } l \geq p_i$$

Now for $p = max\{p_1, p_2\}$ and $l \geq p$ one has

$$\begin{aligned}
\|\{x_n^l - x_n\}\|_{\phi,\mathbb{BC}}^2 &= \frac{1}{2}(\|\{x_{n,i}^l - x_{n,i}\}\|_\phi^2 + \|\{x_{n,i}^l - x_{n,i}\}\|_\phi^2) \\
&< \frac{1}{2}(\epsilon + \epsilon) \\
&= \epsilon.
\end{aligned}$$

Thereby showing that $\{x_n^k\}_{k=1}^\infty$ is a convergent sequence in $l^\phi(\mathbb{BC})$. Therefore $l^\phi(\mathbb{BC})$ equipped with $\|.\|_{\phi,\mathbb{BC}}$ is a Banach space over $\mathbb{C}$. ■

Theorem 2.20. *Let (Ω, Σ, μ) be a σ–finite and purely atomic measure space with atoms $\{A_n\}$ with measure $\mu(A_n) = a_n > 0$ for any $n \in \mathbb{N}$. Let $T : \Omega \mapsto \Omega$ be a measurable transformation with $T(\Omega) = \Omega$ and $b_n = \dfrac{\mu \circ T^{-1}(A_n)}{\mu(A_n)}$. Then C_T is bounded from $l^\phi(\mathbb{BC}, \{a_n\})$ into itself if and only if for any $\{x_n\}$ in $l^\phi(\mathbb{BC}, \{a_n\})$, there exist scalars λ_1 and λ_2 such that*

$$\sum_{n=1}^{\infty}(\phi(\lambda_1 x_{n,1}) + \phi(\lambda_2 x_{n,2}))b_n < \infty \tag{2.8}$$

where $x_n = x_{n,1}e + x_{n,2}e^\dagger$.

Proof. Suppose $C_T : l^{\phi}(\mathbb{BC}, \{a_n\}) \mapsto l^{\phi}(\mathbb{BC}, \{a_n\})$ is bounded. Therefore there exists a $k > 0$ such that

$$\|C_T(\{x_n\})\|_{\phi,\mathbb{BC}} \leq k\|\{x_n\}\|_{\phi,\mathbb{BC}} \tag{2.9}$$

for every $\{x_n\} \in l^{\phi}(\mathbb{BC}, \{a_n\})$. Now set x_n^1 and x_n^2 for $x_n = x_{n,1}e + x_{n,2}e^{\dagger} \in \mathbb{BC}$ as

$$x_n^1 = x_{n,1}e + 0e^{\dagger}$$
$$x_n^2 = 0e + x_{n,2}e^{\dagger}.$$

Then clearly $\{x_n^1\}$ and $\{x_n^2\}$ belong to $l^{\phi}(\mathbb{BC}, \{a_n\})$ whenever $\{x_n\} \in l^{\phi}(\mathbb{BC}, \{a_n\})$ and

$$x_{n,1}^1 = x_{n,1}, \qquad x_{n,1}^1 = 0;$$
$$x_{n,2}^2 = 0, \qquad x_{n,2}^2 = x_{n,2}.$$

Therefore, by (2.9), if $\{x_n\} \in l^{\phi}(\mathbb{BC}, \{a_n\})$ we have

$$\|C_T(\{x_n^i\})\|_{\phi,\mathbb{BC}} \leq \|\{x_n^i\}\|_{\phi,\mathbb{BC}}$$

for $i = 1, 2$. Therefore, $\|\{x_{T(n),1}^i\}\|_{\phi} \leq \sqrt{k}\|\{x_{n,1}^i\}\|_{\phi}$. Consequently we have $C_T : l^{\phi}(\{a_n\}) \mapsto l^{\phi}(\{a_n\})$ is bounded. Therefore, by Theorem 2.6 in [4], we have $\lambda_i > 0$ such that

$$\sum_{n=1}^{\infty}(\lambda_i x_{n,i})b_n < \infty \tag{2.10}$$

for $i = 1, 2$. Now (2.10) easily yields (2.8).

Conversely suppose that (2.8) holds. Then there exists $\lambda_i > 0$ such that

$$\sum_{n=1}^{\infty}(\lambda_i x_{n,i})b_n < \infty \tag{2.11}$$

for $i = 1, 2$ and $\{x_n = x_{n,1}e + x_{n,2}e^{\dagger}\} \in l^{\phi}(\mathbb{BC}, \{a_n\})$. Again by Theorem 2.6 in [4], $C_T : l^{\phi}(\{a_n\}) \mapsto l^{\phi}(\{a_n\})$ is bounded. Therefore there exists $k_i > 0$ such that

$$\|\{x_{T(n),i}\}\|_{\phi} \leq k_i\|\{x_{n,i}\}\|_{\phi}.$$

Therefore one has

$$\begin{aligned}\|C_T(\{x_n\})\|^2_{\phi,\mathbb{BC}} &= \|\{x_{T(n)}\}\|^2_{\phi,\mathbb{BC}}\\ &= \frac{1}{2}(\|\{x_{T(n),1}\}\|^2_{\phi} + \|\{x_{T(n)},2\}\|^2_{\phi})\\ &\leq \frac{k_1}{2}\|\{x_{n,1}\}\|^2_{\phi} + \frac{k_2}{2}\|\{x_{n,2}\}\|^2_{\phi}\\ &= \frac{k}{2}(\|\{x_{n,1}\}\|^2_{\phi} + \|\{x_{n,2}\}\|^2_{\phi})\\ &= k\|\{x_n\}\|^2_{\phi,\mathbb{BC}}\end{aligned}$$

where $k = \max\{k_1, k_2\}$. Therefore C_T is bounded from $l^{\phi}(\mathbb{BC}, \{a_n\})$ into itself. ∎

Definition 2.21. An Orlicz function ϕ is called an N - function if

1. ϕ is continuous.
2. $\phi(x) = 0$ if and only if $x = 0$.
3. $\lim_{x\to 0}\frac{\phi(x)}{x} = 0$.
4. $\lim_{x\to\infty}\frac{\phi(x)}{x} = \infty$.

Theorem 2.22. *If* $\widetilde{L}^{\phi}(\mathbb{BC} = \{f_1e + f_2e^{\dagger} : f_1, f_2 \in \widetilde{L}^{\phi}\}$ *then*

1. $\bigcup\{\widetilde{L}^{\phi}(\mathbb{BC}) : \phi$ *ranges over all N - functions*$\} \subseteq L^1(\mathbb{BC})$.
2. $L^{\infty}(\mathbb{BC}) = \bigcap\{\widetilde{L}^{\phi}(\mathbb{BC}) : \phi$ *ranges over all N - functions*$\}$.

Proof.

1. If $f = f_1e + f_2e^{\dagger} \in \bigcup\{\widetilde{L}^{\phi}(\mathbb{BC}) : \phi$ ranges over all N - functions$\}$, then $f = f_1e + f_2e^{\dagger} \in \widetilde{L}^{\phi}(\mathbb{BC})$ for some ϕ. Therefore f_1 and f_2 are in $\widetilde{L}^{\phi}$ for some ϕ. This implies that f_1 and f_2 are in $L^1(\mu)$. Hence $f = f_1e + f_2e^{\dagger} \in L^1(\mathbb{BC})$. Therefore $\bigcup\{\widetilde{L}^{\phi}(\mathbb{BC}) : \phi$ ranges over all N - functions$\} \subseteq L^1(\mathbb{BC})$.
2. Let $f = f_1e + f_2e^{\dagger} \in L^{\infty}(\mathbb{BC})$, where f_1 and f_2 are in $L^{\infty}(\mu)$. Therefore f_1 and f_2 are in $\widetilde{L}^{\phi}$ for all ϕ. This implies that $f = f_1e + f_2e^{\dagger} \in \widetilde{L}^{\phi}(\mathbb{BC})$ for all ϕ. Hence $f = f_1e + f_2e^{\dagger} \in \bigcap\{\widetilde{L}^{\phi}(\mathbb{BC}) : \phi$ ranges over all N - functions$\}$. Conversely if $f = f_1e + f_2e^{\dagger} \in \bigcap\{\widetilde{L}^{\phi}(\mathbb{BC}) : \phi$ ranges over all N - functions$\}$, then $f = f_1e + f_2e^{\dagger} \in$

$\widetilde{L}^{\phi}(\mathbb{BC})$ for all ϕ. Therefore f_1 and f_2 are in $\widetilde{L}^{\phi}$ for all ϕ. This implies that f_1 and f_2 are in $L^{\infty}(\mu)$. Hence $f = f_1 e + f_2 e^{\dagger} \in L^{\infty}(\mathbb{BC})$. Therefore $L^{\infty}(\mathbb{BC}) = \bigcap\{\widetilde{L}^{\phi}(\mathbb{BC}) :$ ϕ ranges over all N - functions$\}$.

■

Theorem 2.23. *Let $f_n = f_{n,1}e + f_{n,2}e^{\dagger}, n \geq 1$ be a sequence in $L^{\phi}(\mathbb{BC})$ such that $f_n \to f = f_1 e + f_2 e^{\dagger}$. Then $\|f\|_{\mathbb{BC}} \leq \underline{\lim}\ _{n\to\infty}\|f_n\|_{\mathbb{BC}}$.*

Proof. The result follows immediately from

$$\begin{aligned} 2\|f\|_{\mathbb{BC}}^2 &= \|f_1\|_{\Omega}^2 + \|f_2\|_{\Omega}^2 \\ &\leq \underline{\lim}\ _{n\to\infty}\|f_{n,1}\|_{\Omega}^2 + \underline{\lim}\ _{n\to\infty}\|f_{n,2}\|_{\Omega}^2 \\ &\leq \underline{\lim}\ _{n\to\infty}(\|f_{n,1}\|_{\Omega}^2 + \|f_{n,2}\|_{\Omega}^2) \\ &= 2\ \underline{\lim}\ _{n\to\infty}\|f_n\|_{\mathbb{BC}}^2. \end{aligned}$$

■

Theorem 2.24. *Let $\Omega = (\Omega, \Sigma, \mu)$ be a σ-finite complete measure space and $T : \Omega \mapsto \Omega$ be a non singular measurable transformation. Also suppose that ϕ satisfies Δ_2 condition. Then $C_T : L^{\phi}(\mathbb{BC}) \mapsto L^{\phi}(\mathbb{BC})$ is bounded if and only if there is a constant $m > 0$ such that $\mu \circ T^{-1}(A) \leq m\mu(A)$ for each A such that $\mu(A) < \infty$.*

Proof. First suppose that there is a constant $m > 0$ such that $\mu \circ T^{-1}(A) \leq m\mu(A)$ for each A such that $\mu(A) < \infty$. Therefore there exists a constant k such that

$$\begin{aligned} 2\|C_T f\|_{\mathbb{BC}}^2 &= 2\|C_T f_1 e + C_T f_2 e^{\dagger}\|_{\mathbb{BC}}^2 \\ &= \|C_T f_1\|_{\Omega}^2 + \|C_T f_2\|_{\Omega}^2 \\ &\leq k\|f_1\|_{\Omega}^2 + k\|f_2\|_{\Omega}^2 \\ &= k\ 2\ \|f\|_{\mathbb{BC}}^2, \end{aligned}$$

Therefore $\|C_T f\|_{\mathbb{BC}} \leq \sqrt{k}\|f\|_{\mathbb{BC}}$. Hence C_T is bounded. Conversely suppose that C_T is bounded on $L^{\phi}(\mathbb{BC})$. Then it follows from Remark 2.7 that C_T is bounded on $L^{\phi}(\Omega)$ and therefore the result follows from [4]. ■

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