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MARTINGALES IN $\mathbb{D}$-MODULE VALUED L^p-SPACES

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Abstract. In this paper, we introduce the concept of $\mathbb{D}$-module valued L^p-spaces.We generalised the concept of conditional expectation on classical L^p-spaces to the concept of conditional expectation on $\mathbb{D}$-module valued L^p-spaces. Finally the concept of martingales in these spaces is introduced.

Keywords. $\mathbb{D}$-measure space, $\mathbb{D}$-random variable, conditional expectation, martingales.

1. Introduction

The work is essentially based on the book of M.M.Rao [11]. Let us define the set of extended hyperbolic numbers $\bar{\mathbb{D}}$ as $\bar{\mathbb{D}} = \left\{z = \alpha e + \beta e^{\dagger} | \alpha, \beta \in \bar{\mathbb{R}}\right\}$, and the set of non negative extended hyperbolic numbers

$$\bar{\mathbb{D}}^{+} = \left\{z = \alpha e + \beta e^{\dagger} | \alpha, \beta \in \bar{\mathbb{R}}^{+}\right\},$$

where $\bar{\mathbb{R}}$ is the set of extended real numbers and $\bar{\mathbb{R}}^{+}$ is the set of non negative extended real numbers.If $z_1, z_2 \in \bar{\mathbb{D}}$, then $z_1 + z_2$, $z_1 z_2$ and $0z_1$ may be undefined unless $z_1, z_2 \in \mathbb{D}$. Let (Ω, Σ, μ) be a measure space and $\mathfrak{B} \subset \Sigma$ a σ-subalgebra such that $\mu/\mathfrak{B}$ is localizable.If $f \colon \Omega \to \bar{\mathbb{R}}$ is any measurable function such that f^{+} or f^{-} is μ-integrable, then recall that any $\mathfrak{B}$- measurable function $\tilde{f} \colon \Omega \to \bar{\mathbb{R}}$ satisfying the system of equations

$$\int_B f \, d\mu = \int_B \tilde{f} \, d\mu/\mathfrak{B}, \; B \in \mathfrak{B},$$

is called a version of conditional expectation of f given $\mathfrak{B}$, and is denoted by $E_{\mathfrak{B}}(f) = \tilde{f}$ see [12] Let $f \colon \Omega \to \bar{\mathbb{D}}^{+}$ be a $\mathbb{D}$- measurable function on a $\mathbb{D}$- measure space $(\Omega, \Sigma, \mu_{\mathbb{D}})$ and $\mathfrak{B} \subset \Sigma$ be a σ-subalgebra such that $\mu_{\mathbb{D}}/\mathfrak{B}$ is localizable.Then $f = ef_1 + e^{\dagger} f_2$, where $f_i \colon \Omega \to \bar{\mathbb{R}}^{+}, i = 1, 2$ are real valued measurable functions on $(\Omega, \Sigma, \mu_{\mathbb{D}})$. The idempotent components $\mu_i/\mathfrak{B}, i = 1, 2$ of $\mu_{\mathbb{D}}/\mathfrak{B}$ are localizable.

2. $\mathbb{D}$-module valued L^p-spaces

If $E_{\mathfrak{B}}(f_i), i = 1, 2$ are conditional expectations of $f_i, i = 1, 2$ relative to $\mathfrak{B}$

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then we call $E_{\mathfrak{B}}(f) = eE_{\mathfrak{B}}(f_1) + e^{\dagger}E_{\mathfrak{B}}(f_2)$ the conditional expectation of f relative to $\mathfrak{B}$. We denote by $L^p(\Omega, \Sigma, \mu_{\mathbb{D}})$, the set of all $\mathbb{D}$-measurable functions f on Ω such that $|f|_k^p$ is $\mathbb{D}$-lebesgue integrable.This set turns out to be a Banach $\mathbb{D}$-module under the operations of pointwise addition and scalar multiplication equipped with hyperbolic norm which can be decomposed as $L^p(\Omega, \Sigma, \mu_{\mathbb{D}}) = eL^p(\Omega, \Sigma, \mu_1) + e^{\dagger}L^p(\Omega, \Sigma, \mu_2)$, where $L^p(\Omega, \Sigma, \mu_1)$ and $L^p(\Omega, \Sigma, \mu_2)$ are classical spaces of equivalence classes of real valued measurable functions whose pth power is $\mathbb{D}$- Lebesgue integrable. The properties exhibited by conditional expectations of real valued measurable functions can be lifted to the expectations of $\mathbb{D}$-measurable functions. Let $X = eX_1 + e^{\dagger}X_2$ be a Banach $\mathbb{D}$- module equipped with hyperbolic norm and a Schauder basis $\{u_i\}_{i=1}^{\infty}$ and $(\Omega, \Sigma, \mu_{\mathbb{D}})$ be a finite $\mathbb{D}$- measure space. Then every $f\colon \Omega \to X$ can be written as $f(w) = \Sigma_{i=1}^{\infty} f_i(w)u_i$. If each f_i is $\mathbb{D}$- measurable function on Ω, then we say that f is measurable. For $1 \le p < \infty$, the set of all measurable functions $f\colon \Omega \to X$ such that $||f||_{\mathbb{D}} \in L^p(\Omega, \Sigma, \mu_{\mathbb{D}})$ is denoted by $L^p(\mu_{\mathbb{D}}, X)$. That is,

$$L^p(\mu_{\mathbb{D}}, X) = \{f : \Omega \to X \mid f \text{ is measurable and } ||f||_{\mathbb{D}} \in L^p(\Omega, \Sigma, \mu_{\mathbb{D}})\}$$

and it forms a Banach $\mathbb{D}$-module under the operations of pointwise addition and scalar multiplication, where the norm of any element f is given by
$||f||_{L_p(\mu_{\mathbb{D}}, X)} = (\int_{\Omega} ||f||_{\mathbb{D}}^p \, d\mu_{\mathbb{D}})^{\frac{1}{p}}$. This space can be decomposed as

$$L^p(\mu_{\mathbb{D}}, X) = eL^p(\mu_1, X_1) + e^{\dagger}L^p(\mu_2, X_2),$$

where

$$L^p(\mu_i, X_i) = \{f_i : \Omega \to X_i \mid f_i \text{ is measurable and } ||f_i||_i \in L^p(\Omega, \Sigma, \mu_i)\}$$

are Banach spaces with $||f_i||_{L_p(\mu_i, X_i)} = (\int_{\Omega} ||f_i||_i^p \, d\mu_i)^{\frac{1}{p}}$ for each i=1,2.A sequence $\{f_n\}$ converges to f in $L^p(\mu_{\mathbb{D}}, X)$ iff $||f_n - f||_{\mathbb{D}}$ converges to 0 in $L^p(\Omega, \Sigma, \mu_{\mathbb{D}})$.

Theorem 2.1. *Let $(\Omega, \Sigma, \mu_{\mathbb{D}})$ be a $\mathbb{D}$-measure space and $1 \le p < \infty$. Then for each $\epsilon \in \mathbb{D}^+$, $f \in L^p(\mu_{\mathbb{D}}, X)$, there exists a function $h_\epsilon = \Sigma_{i=1}^{\infty} \alpha_i f_i \in L^p(\mu_{\mathbb{D}}, X)$, where each $f_i\colon \Omega \to \mathbb{D}$ is a simple function such that $||f - h_\epsilon||_{L^p(\mu_{\mathbb{D}}, X)} \prec \epsilon$.*

Proof. Let $f = \Sigma_{i=1}^{\infty} f_i u_i \in L^p(\mu_{\mathbb{D}}, X)$ and let $\epsilon \in \mathbb{D}^+$ be given. Then $f_i \in L^p(\Omega, \Sigma, \mu_{\mathbb{D}})$ for each i.Therefore for each i, there exists a simple function $f_{\epsilon_i} \in L^p(\Omega, \Sigma, \mu_{\mathbb{D}})$ such that $||f_i - f_{\epsilon_i}||_{L^p(\Omega, \Sigma, \mu_{\mathbb{D}})} \prec \epsilon(\frac{1}{i(i+1)}) = \epsilon_i$.
Let $h_\epsilon = \Sigma_{i=1}^{\infty} f_{\epsilon_i} u_i$. Then $h \in L^p(\mu_{\mathbb{D}}, X)$

$$\begin{aligned} & ||f - h_\epsilon||_{L^p(\mu_{\mathbb{D}}, X)} \\ & = ||\Sigma_{i=1}^{\infty} (f_i - f_{\epsilon_i}) u_i||_{L^p(\mu_{\mathbb{D}}, X)} \\ & \preceq \Sigma_{i=1}^{\infty} ||f_i - f_{\epsilon_i}||_{L^p(\Omega, \Sigma, \mu_{\mathbb{D}})} ||u_i||_{\mathbb{D}} \\ & = \Sigma_{i=1}^{\infty} ||f_i - f_{\epsilon_i}||_{L^p(\Omega, \Sigma, \mu_{\mathbb{D}})} \\ & \prec \Sigma_{i=1}^{\infty} \epsilon_i \\ & = \Sigma_{i=1}^{\infty} \epsilon(\frac{1}{i(i+1)} = \epsilon. \end{aligned} \tag{2.1}$$

□

Theorem 2.2. *(Dominated Convergence Theorem) Let $(\Omega, \Sigma, \mu_{\mathbb{D}})$ be a $\mathbb{D}$-measure space and $\{f_n\}$ be a sequence of X-valued measurable functions on Ω such that $\lim\limits_{n \to \infty} f_n(w) = f(w), \forall w \in \Omega$. If there exists a $\mathbb{D}$-valued lebesgue integrable measurable function g on Ω such that $||f_n(w)||_{\mathbb{D}} \preceq g(w), n = 1, 2, 3, \ldots, w \in \Omega$, then $f \in$*

$L^p(\mu_{\mathbb{D}}, X)$ *and*

$$\lim_{n\to\infty} \int_\Omega f_n \, d\mu_{\mathbb{D}} = \int_\Omega f \, d\mu_{\mathbb{D}}.$$

Proof. Take $g_n = ||f_n - f||_{\mathbb{D}}$ and dominating function as 2g. The proof follows by applying the scalar Dominated Convergence Theorem to the sequence $\{g_n\}$. □

3. Conditional Expectation

Let $(\Omega, \Sigma, \mu_{\mathbb{D}})$ be a finite measure space and $\mathfrak{B}$ be a sub σ-algebra of Σ. If $\lambda\colon \mathfrak{B} \to X$ is countably additive set function, then we can write $\lambda = \Sigma_{i=1}^{\infty} \lambda_{\mathbb{D}}^i u_i$, where each $\lambda_{\mathbb{D}}^i\colon \mathfrak{B} \to \mathbb{D}$ is countably additive. Let $f\colon \Omega \to X$ be given by $f(w) = \Sigma_{i=1}^{\infty} f_i(w) u_i$, where each $f_i\colon \Omega \to \mathbb{D}$ and further suppose that $\int_\Omega ||f(x)||_{\mathbb{D}} \, d\mu_{\mathbb{D}} \in \mathbb{D}$. Then $\lambda(E) = \int_E f(w) \, d\mu_{\mathbb{D}}$ defines a X valued set function on Σ and so it can be written as $\lambda(E) = \Sigma_{i=1}^{\infty} \lambda_{\mathbb{D}}^i(E) u_i$, where $\lambda_{\mathbb{D}}^i(E) = \int_E f_i(w) \, d\mu_{\mathbb{D}}$ for each i.Then we have the following definition.

Definition 3.1. If $f(w) = \Sigma_{i=1}^{\infty} f_i(w) u_i$ is integrable,where each $f_i\colon \Omega \to \mathbb{D}$ is $\mathbb{D}$-measurable and $g(w) = \Sigma_{i=1}^{\infty} E^{\mathfrak{B}}(f_i)(w) u_i$, then we call g the conditional expectation of f given $\mathfrak{B}$ and we write $g = E^{\mathfrak{B}}(f)$.

Remark 3.2. If$f(w) = \Sigma_{i=1}^{\infty} f_i(w) u_i$ is measurable, then each f_i is $\mathbb{D}$-measurable for $\mathfrak{B}$ and $\int_B f_i \, d\mu_{\mathbb{D}} = \int_B E^{\mathfrak{B}}(f_i) \, d\mu_{\mathbb{D}}, \forall B \in \mathfrak{B}$. This gives
$\int_B f \quad d\mu_{\mathbb{D}} = \Sigma_{i=1}^{\infty} u_i \int_B f_i \quad d\mu_{\mathbb{D}} = \Sigma_{i=1}^{\infty} u_i \int_B E^{\mathfrak{B}}(f_i) \, d\mu_{\mathbb{D}} = \int_{\mathfrak{B}} \Sigma_{i=1}^{\infty} u_i E^{\mathfrak{B}}(f_i) = \int_{\mathfrak{B}} E^{\mathfrak{B}}(f) \, d\mu_{\mathbb{D}}$.

We can decompose each $E^{\mathfrak{B}}(f_i)$ as $E^{\mathfrak{B}}(f_i) = eE^{\mathfrak{B}}(f_i^1) + e^{\dagger} E^{\mathfrak{B}}(f_i^2)$, where each $f_i = ef_i^1 + e^{\dagger} f_i^2$. and so

$$\begin{aligned} E^{\mathfrak{B}}(f) &= e\Sigma_{i=1}^{\infty} E^{\mathfrak{B}}(f_i^1)(w) u_i \\ &\quad + e^{\dagger} \Sigma_{i=1}^{\infty} E^{\mathfrak{B}}(f_i^2)(w) u_i \\ &= eE^{\mathfrak{B}}(f^1) + e^{\dagger} E^{\mathfrak{B}}(f^2), \end{aligned} \tag{3.1}$$

where f^j is X_j valued measurable and integrable function and $\left\{u_i^j\right\}_{i=1}^{\infty}$ is schauder basis of X_j for each j=1,2.

Lemma 3.3. *If* $f_i = \Sigma_{j=1}^{\infty} f_i^j u_i^j \in L^p(\mu_i, X_i)$ *and* $\Sigma_{i=1}^{\infty} ||f_i^j|| < \infty$, *then* $E^{\mathfrak{B}}(f_i) \in L^p(\mu_i, X_i)$ *for each i=1,2.*

Proof. If $f_i = \Sigma_{j=1}^{\infty} f_i^j u_i^j$, then $E^{\mathfrak{B}}(f_i) = \Sigma_{j=1}^{\infty} E^{\mathfrak{B}}(f_i^j) u_i^j$. Therefore

$$\begin{aligned} ||E^{\mathfrak{B}}(f_i)||_{L^P(\mu_i, X_i)} &= ||\Sigma_{j=1}^{\infty} E^{\mathfrak{B}}(f_i^j) u_i^j||_{L^P(\mu_i, X_i)} \\ &\leq \Sigma_{j=1}^{\infty} ||E^{\mathfrak{B}}(f_i^j)||||u_i^j||_i \\ &= \Sigma_{j=1}^{\infty} ||E^{\mathfrak{B}}(f_i^j)|| \\ &\leq \Sigma_{j=1}^{\infty} ||f_i^j|| < \infty. \end{aligned}$$

Hence $E^{\mathfrak{B}}(f) \in L^p(\mu_i, X_i)$ for each i=1,2. □

Theorem 3.4. *If* $f = \Sigma_{i=1}^{\infty} f_i \quad u_i \in L^p(\mu_{\mathbb{D}}, X)$ *and* $\Sigma_{i=1}^{\infty} ||f_i|| \in \mathbb{D}$, *then* $E^{\mathfrak{B}}(f) \in L^p(\mu_{\mathbb{D}}, X)$.

Proof. We have $\Sigma_{i=1}^{\infty} ||f_i|| = e\Sigma_{i=1}^{\infty} ||f_i^1|| + e^{\dagger} \Sigma_{i=1}^{\infty} ||f_i^2|| \in \mathbb{D}$. Therefore $\Sigma_{i=1}^{\infty} ||f_i^j|| < \infty$ for each i=1,2. and so by Lemma 3.3, $E^{\mathfrak{B}}(f_i) \in L^p(\mu_i, X_i)$ for each i=1,2. Hence $E_{\mathfrak{B}}(f) = eE_{\mathfrak{B}}(f^1) + e^{\dagger} E_{\mathfrak{B}}(f^2) \in eL^p(\mu_1, X_1) + e^{\dagger} L^p(\mu_2, X_2) = L^p(\mu_{\mathbb{D}}, X)$. □

The operator $E^{\mathfrak{B}}\colon L^1(\mu_{\mathbb{D},X}) \to L^1(\mu_{\mathbb{D},X})$ satisfies the following properties:

(i) $E^{\mathfrak{B}}$ is linear transformation, i.e, $E^{\mathfrak{B}}(\alpha \quad f + \beta \quad g) = \alpha E^{\mathfrak{B}}(f) +$

$\beta E^{\mathfrak{B}}(g), \forall \alpha, \beta \in \mathbb{D}$ and $f, g \in L^1(\mu_{\mathbb{D}}, X)$.

(ii) $E^{\mathfrak{B}}$ is a contraction, i.e, $||E^{\mathfrak{B}}(f)||_{L^1} \preceq ||f||_{L^1}$

(iii) $E^{\mathfrak{B}}(E^{\mathfrak{B}}(f)) = E^{\mathfrak{B}}(f), \forall f \in L^1(\mu_{\mathbb{D}}, X)$.

(iv) If $\mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \Sigma$ are σ-algebras and $\mu_{\mathbb{D}}/\mathfrak{B}_{\mathfrak{i}}$ are localizable, then

$$E^{\mathfrak{B}_1}(E^{\mathfrak{B}_2}(f)) = E^{\mathfrak{B}_2}(E^{\mathfrak{B}_1}(f)) = E^{\mathfrak{B}_1}(f).$$

(v) If $\mathfrak{H} \subset \mathfrak{B} \subset \Sigma$, then $E^{\mathfrak{H}}(E^{\mathfrak{B}}(f)) = E^{\mathfrak{B}}(f)$.

Proof. (i)

$$\begin{aligned}
&\int_A E^{\mathfrak{B}}(\alpha\ f + \beta\ g) d\ \mu_{\mathbb{D}}/\mathfrak{B} \\
&= \int_A (\alpha\ f + \beta\ g) d\ \mu_{\mathbb{D}} \\
&= \alpha \int_A f d\ \mu_{\mathbb{D}} + \beta \int_A g d\ \mu_{\mathbb{D}} \\
&= \alpha \int_A E^{\mathfrak{B}}(f) d\ \mu_{\mathbb{D}}/\mathfrak{B} \\
&\quad + \ \beta \int_A E^{\mathfrak{B}}(g) d\ \mu_{\mathbb{D}}/\mathfrak{B} \\
&= \int (\alpha\ E^{\mathfrak{B}}(f) + \beta\ E^{\mathfrak{B}}(g)) d\ \mu_{\mathbb{D}}/\mathfrak{B}.
\end{aligned}$$

Hence,

$$E^{\mathfrak{B}}(\alpha\ f + \beta\ g) = \alpha E^{\mathfrak{B}}(f) + \beta E^{\mathfrak{B}}(g).$$

(ii)

$$\begin{aligned}
\int_A E^{\mathfrak{B}}(f.g) d\ \mu_{\mathbb{D}}/\mathfrak{B} &= \int_A (f.g) d\ \mu_{\mathbb{D}} \\
&= f. \int_A f d\ \mu_{\mathbb{D}} \\
&= f.E^{\mathfrak{B}}(g).
\end{aligned}$$

□

4. Martingales

Definition 4.1. Let $(\Omega, \Sigma, \mu_{\mathbb{D}})$ be a $\mathbb{D}$-measure space with finite subset property and Σ_n be an increasing sequence of σ-subalgebras of Σ such that $\mu_{\mathbb{D}}/\Sigma_n, n \geq 1$ is localizable. If $\{f_n : n \geq 1\}$ is a sequence in $L^p(\mu_{\mathbb{D}}, X)$ such that f_n is measurable for $\Sigma_n, n \geq 1$, then $\{(f_n, \Sigma_n) : n \geq 1\}$ is called a martingale if for each $n \geq 1$,

$$E^{\Sigma_n}(f_{n+1}) = f_n. \tag{4.1}$$

It is called a supermartingale if $=$ is replaced by $\leq$ and submartingale if $=$ is replaced by $\geq$ there. We denote the martingale of above form by $\{f_n, \Sigma_n : n \geq 1\}$ to display both the functions and $\sigma - subalgebras$.

Example 4.2. Let $f \in L^p(\nu_{\mathbb{D}}, X)$ and $\{\Sigma_n\}$ be an increasing sequence of σ-subalgebras of Σ. If $f_n = E^{\Sigma_n}(f)$, then the sequence $\{f_n, \Sigma_n : n \geq 1\}$ is a martingale in $L^p(\nu_{\mathbb{D}}, X)$.

Remark 4.3. We can write every measurable function $f\colon \Omega \to X$ as $f(w) = \Sigma_{i=1}^{\infty} f_i(w) u_i$, where each $f_i\colon \Omega \to \mathbb{D}$ is $\mathbb{D}$-measurable and further each f_i can be written as $f_i = e f_i^1 + e^{\dagger} f_i^2$ such that f_i^j is real measurable for j=1,2. Therefore $f(w) = e f^1(w) + e^{\dagger} f^2(w)$, where $f^i\colon \Omega \to X_i$ is measurable for i=1,2. Thus every martingale $\{(f_n, \Sigma_n) : n \geq 1\}$ can be decomposed as $\{e(f_n^1, \Sigma_n) : n \geq 1\} + e^{\dagger} \{(f_n^2, \Sigma_n) : n \geq 1\}$, where $\{f_n^j\}$ is a sequence in $L^p(\mu_j, X_j)$ such that μ_j^n is localizable and f_n^j is measurable for $\Sigma_n, n \geq 1$ for each j=1,2. Also by using 3.1,we have $E^{\Sigma_n}(f_{n+1}^j) = f_n, \ \forall n \geq 1$, j=1,2.Thus $\{(f_n^j, \Sigma_n) : n \geq 1\}$ is a martingale for each j=1,2.Hence the study of X-valued martingales is equivalent to the study of a pair of X_j-valued martingales for j=1,2.

Proposition 4.4. *A sequence $\{f_n, \Sigma_n\}_{n\geq 1}$ is a martingale in $L^p(\nu_{\mathbb{D}}, X)$ iff the sequence $\{f_n^j, \Sigma_n : n \geq 1\}$ is a martingale in $L^p(\nu_j, X_j)$, j=1,2.*

Proof. First suppose that the sequence $\{f_n, \Sigma_n : n \geq 1\}$ is a martingale in $L^p(\nu_{\mathbb{D}}, X)$. For each $n \geq 1$, we can write $f_n = ef_n^1 + e^{\dagger_2} f_n^2$ and $\nu_{\mathbb{D}}/\Sigma_n = e\nu_1/\Sigma_n + e^{\dagger_2}\nu_2/\Sigma_n$, where $\{f_n^j\}$ is a sequence in $L^p(\nu_j, X_j)$ and ν_j/Σ_n is localizable, j=1,2. Now by (4.1), we have

$$\begin{aligned} E^{\Sigma_n}(f_{n+1}) &= eE^{\Sigma_n}(f_{n+1}^1) + e^{\dagger_2}E^{\Sigma_n}(f_{n+1}^2) \\ &= f_n \\ &= ef_n^1 + e^{\dagger_2} f_n^2, n \geq 1, \end{aligned}$$

which gives $E^{\Sigma_n}(f_{n+1}^j) = f_n^j$, j=1,2. Thus $\{f_n^j, \Sigma_n : n \geq 1\}$ is a martingale in $L^p(\nu_j, X_j)$, j=1,2.

Conversely, suppose that $\{f_n^j, \Sigma_n : n \geq 1\}$ is a martingale in $L^p(\nu_j, X_j)$, j=1,2. Then $f_n = ef_n^1 + e^{\dagger_2} f_n^2$ and so $\{f_n, \Sigma_n : n \geq 1\}$ is a sequence in $L^p(\nu_{\mathbb{D}}, X)$ such that f_n is Σ_n-measurable and

$$\begin{aligned} E^{\Sigma_n}(f_{n+1}) &= eE^{\Sigma_n}(f_{n+1}^1) + e^{\dagger_2}E^{\Sigma_n}(f_{n+1}^2) \\ &= ef_n^1 + e^{\dagger_2} f_n^2 \\ &= f_n, n \geq 1. \end{aligned}$$

Thus $\{f_n, \Sigma_n : n \geq 1\}$ is a martingale. □

Proposition 4.5. *Let $\{f_n, \Sigma_n : n \geq 1\}$ be a martingale in $L^p(\nu_{\mathbb{D}}, X)$. Then the sequence $\left\{\int_E f_n \, d\nu_{\mathbb{D}}, n \geq 1\right\}$ is convergent for every $E \in \bigcup_{n=1}^{\infty} \Sigma_n$.*

Proof. Let $\{(f_n, \Sigma_n) : n \geq 1\}$ be a martingale and $E \in \cup_{n=1}^{\infty}\Sigma_n$. Since $\{\Sigma_n, n \geq 1\}$ is a monotonically increasing sequence of σ-subalgebras of Σ and $E \in \cup_{n=1}^{\infty}\Sigma_n$, there exist $n_0 \in \mathbb{N}$ for which $E \in \Sigma_n$, for all $n \geq n_0$. Thus for $n \geq n_0$, we get

$$\int_E f_n \, d\nu_{\mathbb{D}} = \int_E E^{\Sigma_{n_0}}(f_n) d\nu_{\mathbb{D}} = \int_E f_{n_0} \, d\nu_{\mathbb{D}}$$

as $E^{\Sigma_{n_0}}(f_n) = f_{n_0}$. Hence the sequence $\left\{\int_E f_n \, d\nu_{\mathbb{D}}, n \geq 1\right\}$ is eventually constant and so convergent.

□

This property is very useful in the study of norm convergent martingales.

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